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Crossing Relations for Helicity Amplitudes*

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Crossing relations for helicity amplitudes for particles of arbitrary spin are formulated without recourse to the introduction of scalar amplitudes. The basic assumption is that the amplitudes are simply related by analytic continuation; the path of continuation is carefully specified. The relations are given a simple geometrical interpretation. The relation between $\pi N \rightarrow \pi N$ and $\pi\pi \rightarrow N\bar{N}$ obtained in this way agrees with that obtained by direct elimination of scalar amplitudes.

I. INTRODUCTION

The most common applications of crossing relations involve particles of spin 0 or at most ½. The customary Dirac formalism allows one to express the reaction amplitude in terms of so-called scalar amplitudes ("A" and "B" in the case of π - N scattering) and the crossing-relation then simply states that analytic continuation of a scalar amplitude from the physical region of a channel to that of a "crossed" channel yields the corresponding scalar amplitude in the crossed channel. The introduction of scalar amplitudes is not a simple matter in the general case, so that it would be technically advantageous[1] to formulate the crossing relations in terms of some other amplitudes, which are more easily generalized, for example helicity amplitudes (*1*, *2*).

A crossing relation for helicity amplitudes for a simple case, such as πN scattering versus $N\bar{N}$ annihilation into two pions, can, of course, be obtained indirectly by elimination of the scalar amplitudes A and B from the equations connecting A and B to the helicity amplitudes $F_{\lambda\mu}$ and $G_{\lambda\mu}$ for the two crossed reactions. But the relation obtained is not very transparent at first sight. How can it be generalized?

Recently we arrived at a very simple geometrical interpretation of these relations, which suggests an obvious generalization. Essentially the same interpretation has been arrived at independently and apparently somewhat earlier by M. S. Marinov and V. I. Roginskii (*3*) and by Ya. A. Smorodinsky (*4*). The results of these authors, however, are only similar to but not identical with ours.

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[1] We are indebted to Prof. M. Goldberger for repeatedly drawing our attention to this question.

In particular, the formulas derived by these authors do not seem to agree with those obtained by the elementary "indirect" method for πN scattering, and must therefore be incorrect. We believe that this is due to certain complications arising from analytic continuation, complications which are not discussed in either of the above-mentioned papers. We hope, therefore, that the following remarks will contribute to the further elucidation of this problem.

II. AN EXAMPLE: πN SCATTERING

Let us first briefly recall the results of the calculation via the scalar amplitudes. The πN scattering amplitude is given by

$$\begin{gathered} \bar{u}(p_2)Tu(p_1) \\ T = -A + \tfrac{1}{2}\mu^{-1}i\gamma\cdot(q_1 + q_2)B. \end{gathered} \tag{1}$$

The notation is standard (*5*), except that the usual B is replaced by B/μ. The helicity amplitudes G_{++} and G_{+-}, say, are then obtained by an appropriate choice of the Dirac spinors $u(p_1)$ and $u(p_2)$. The formulas of this section are based on the phase conventions of ref. *1*. Note that according to Eq. (13) of (*1*), there is a factor $(-1)^{s_2-\lambda_2}$ in the definition of the helicity state for "particle 2." In the following equations the pion in both the initial and final state is taken as "particle 2," so that the factor is avoided. One has, ignoring an irrelevant overall phase factor,

$$G_{++} = \cos\left(\frac{\theta_s}{2}\right)\left(A + \frac{k^2 + \epsilon\omega}{m\mu}B\right), \qquad G_{+-} = \sin\left(\frac{\theta_s}{2}\right)\left(\frac{\epsilon}{m}A + \frac{\omega}{\mu}B\right), \tag{2}$$

where k and θ_s are c.m. momentum and scattering angle, $\epsilon = (m^2 + k^2)^{1/2}$ and $\omega = (\mu^2 + k^2)^{1/2}$ are c.m. energies of nucleon and meson respectively. For the relation of these variables to the Mandelstam variables s, t, u (or $\bar{s}$) and related notations we refer the reader to the papers of W. R. Frazer and J. R. Fulco (*6*). With the abbreviation

$$S^2 = [s - (m + \mu)^2][s - (m - \mu)^2] \tag{3}$$

we have

$$k^2 = S^2/4s; \qquad \sin(\theta_s/2) = (-st)^{1/2}/S \tag{4}$$

$$\cos(\theta_s/2) = (S^2 + st)^{1/2}/S = [(m^2 - \mu^2)^2 - su]^{1/2}/S \tag{5}$$

$$\epsilon = (s + m^2 - \mu^2)/(4s)^{1/2}; \qquad \omega = (s - m^2 + \mu^2)/(4s)^{1/2}. \tag{6}$$

These formulas are to be used in the physical region for the s-channel (πN scattering). In this region the square roots in (4)–(6) are all positive by definition. In the t-channel ($\pi\pi \to N\bar{N}$) the helicity amplitudes are given (*6*) by

$$F_{++} = -(p/m)A + (q/\mu)\cos\theta\, B, \qquad F_{+-} = (Eq/m\mu)\cos\theta\, B \tag{7}$$

where:

$$p = (\tfrac{1}{4}t - m^2)^{1/2}, \qquad q = (\tfrac{1}{4}t - \mu^2)^{1/2}, \qquad \cos\theta = (s - u)/4pq, \tag{8}$$

and the functions $A(s, t)$, $B(s, t)$ in the t-channel region are analytic continuations of the corresponding functions in the s-channel region. In these equations, the nucleon in the final state is taken as "particle 2." If A and B have the kind of singularities that are postulated in the Mandelstam representation, the analytic continuation has to go from a point, say, $s = s_i + i\epsilon$, $t = t_i - i\epsilon$, where ϵ is infinitesimal and positive, $s_i > (m + \mu)^2$, $t_i < 0$, $s_i u_i < (m^2 - \mu^2)^2$ to a point $s = s_f - i\epsilon$, $t = t_f + i\epsilon$ where $s_f < 0$ and $s_f u_f > (m^2 - \mu^2)^2$. In these equations the variable u is, of course, $u = 2(m^2 + \mu^2) - s - t$. If we assume, for the sake of simplicity, that u stays real along the trajectory, then in order to reverse the sign of the imaginary parts of s and t, the trajectory must go through a point of the real s, t-plane. If this is the only real point of the trajectory, and if the real point lies within the triangle defined by the inequalities:

$$s < (m + \mu)^2, \qquad u < (m + \mu)^2, \qquad t < 4\mu^2, \tag{9}$$

then we will have insured that the endpoint is still on the "first sheet" where A and B are given by the Mandelstam formulae and have the correct values for the t-channel.

The coefficients of A and B in Eq. (2) have certain singularities (branch-singularities), namely, as can be seen from Eqs. (3)–(6), at $s = 0$, $s = (m \pm \mu)^2$, $t = 0$, and $su = (m^2 - \mu^2)^2$. The trajectory should, of course, avoid these singularities. We may assume that the imaginary part of $s(-t)$ remains small throughout so that the trajectory may be specified for our purposes by drawing a line in the s, t-plane, indicating by a cross the point where the trajectory crosses the real s, t plane. The lines $su = (m^2 - \mu^2)^2$, $s = (m - \mu)^2$, and $t = 0$ divide the triangle (9) into five pieces, see Fig. 1, and depending on where the crossing-point lies, one will get different determinations of the coefficients in Eq. (2) at the final point in the t-channel. Since an over-all change in sign of G_{++} and G_{+-} is unimportant, we have to distinguish only two cases: if the crossing point lies within the hyperbolic segment delimited by $t = 0$ and $su = (m^2 - \mu^2)^2$, then the final values of $\cos(\theta_s/2)$ and $s^{-1/2}\sin(\theta_s/2)$ are pure imaginary and of the same sign. In all other cases, they are pure imaginary and of opposite sign. Thus if we adopt the first alternative, we have

$$\cos(\theta_s/2) = -2ipq\sin\theta/S, \qquad s^{-1/2}\sin(\theta_s/2) = 2iE/S, \tag{10}$$

where $2E = t^{1/2}$ is the total energy in the c.m. system. The minus sign in the expression for $\cos(\theta_s/2)$ is explained in the Appendix. After introducing these

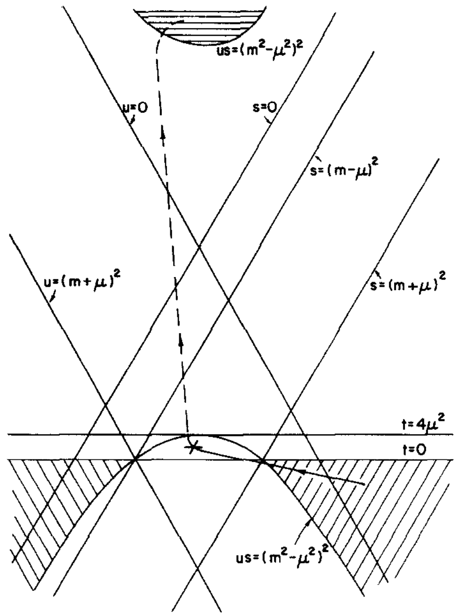

FIG. 1. The s, t plane for $\pi N \to \pi N$ and $\pi\pi \to N\bar{N}$ (drawn for $m = 4\mu$)

values into Eq. (2), we may eliminate A and B from (2) and (7), obtaining

$$\begin{aligned} G_{++} &= (2i/S)[mq \sin\theta F_{++} + E(p - q\cos\theta)F_{+-}], \\ G_{+-} &= (2i/S)[E(p - q\cos\theta)F_{++} - mq\sin\theta F_{+-}], \end{aligned} \tag{11}$$

where, as pointed out before, it is irrelevant which of the two determinations for the square root of (3) is used, provided it is the same in the two equations.[2]

It is easy to verify the identity

$$(mq\sin\theta)^2 + E^2(p - q\cos\theta)^2 = \tfrac{1}{4}S^2 \tag{12}$$

so that, apart from the uninteresting factor i, the transformation matrix between the two sets of helicity amplitudes is an orthogonal matrix. But the meaning of this transformation is not immediately apparent.

Owing to the orthogonal nature of the transformation we are, however, tempted to write Eq. (11), disregarding the factor i, in the form

$$G_{++} = \sin\chi F_{++} + \cos\chi F_{+-}, \qquad G_{+-} = \cos\chi F_{++} - \sin\chi F_{+-} \tag{11'}$$

where χ is determined by

[2] The opposite sign for S in the two equations corresponds, however, to the alternative choice of the crossing point!

$$\tan \chi = \frac{mq \sin \theta}{E(p - q) \cos \theta}. \tag{13}$$

In order to interpret this formula, we must re-examine carefully the process of analytic continuation.

III. GEOMETRICAL INTERPRETATION

In the customary presentation of the crossing relations one rewrites the conservation law for πN scattering

$$q_1 + p_1 = q_2 + p_2 \tag{14}$$

in the form

$$q_1 - q_2 = -p_1 + p_2 \tag{15}$$

and reinterprets $-q_2$ and $-p_1$ as initial four-momentum of a pion and final four-momentum of an antinucleon respectively. It is clear that in this interpretation the values of the four-momenta are *not the same* in Eqs. (14) and (15), since q_2 and p_1 are positive timelike in Eq. (14) and negative timelike in Eq. (15). In fact, the values of Eq. (14) correspond to the initial point of the trajectory of Fig. 1, those of (15) to the final point. Moreover one sees that also the values of q_1 and p_2 have to vary along the trajectory since

$$u = (q_1 - p_2)^2 \tag{16}$$

cannot remain constant. However, this variation of q_1 and p_2 is often disregarded, since in the end the two four-vectors revert to the real positive timelike mass shell. In fact, if the initial point of the trajectory of Fig. 1 lies in the $u < 0$ part of the physical region, we may indeed assume that the final values of q_1 and p_2 are identical with the initial ones. Henceforward we shall make this assumption for simplicity.

Thus in the latter case indicating by primes the values of $q_1, \cdots$ etc. at the end of the trajectory, we may write

$$q_1' = q_1, \quad p_2' = p_2, \quad q_2' = -Q_2, \quad p_1' = -P_1, \tag{17}$$

where $q_1 Q_2$ are the pion momenta in the initial state, and $p_2 P_1$ are the nucleon and antinucleon momenta in the final state of the reaction in the t-channel. It should be noted that P_1 and Q_2 must be different from p_1 and q_2 in Eq. (14). In this respect the usual notation is apt to lead to confusion.

We may clearly assume that at every stage of the analytic continuation the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}_1$, $\mathbf{q}_2$ (whether real or complex) lie in the xz plane, since this gives us sufficient freedom to vary s and t at will. This assumption avoids phase

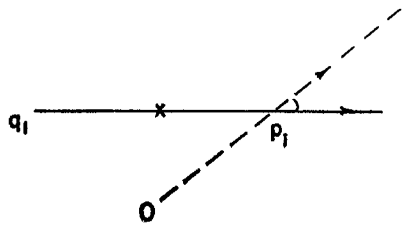

FIG. 2. Velocity space diagram. The ends of the segments p_1 and q_1 represent the velocity points of p_1 and q_1 ; X represents the velocity point of the c.m. of p_1 and q_1 ; 0 is an arbitrary point. The arrows indicate the directions in which the spin components are measured.

factors connected with the azimuthal variable ϕ,[3] and furthermore it means that the y axis is not affected in any of the Lorentz transformations we shall encounter in the following discussion.

Let us now introduce, in addition to the customary helicity amplitudes $G_{\mu\lambda}(s, t)$ so far employed, the notion of "generalized" helicity amplitudes $G_{\mu\lambda}(p_2q_2 ; p_1q_1)$ by which we mean matrix elements of the scattering matrix T between helicity states satisfying condition (14) but not subject to the c.m. condition: $\mathbf{p}_1 + \mathbf{q}_1 = 0$. The phases of the nucleon helicity states are defined by Eq. (6) of (*1*), with $\phi = 0$ and θ unrestricted. The discontinuity at the "south pole" causes no trouble since the state obtained with $\theta = \pi$ is simply $(-1)^{2s}$ times the state with $\theta = -\pi$, independent of the helicity. Let $\boldsymbol{\beta}$ be the velocity of the center of mass

$$\boldsymbol{\beta} = (\mathbf{p}_1 + \mathbf{q}_1)/(\epsilon_1 + \omega_1) \tag{18}$$

where $\epsilon_1 = (m^2 + \mathbf{p}_1^2)^{1/2}$, $\omega_1 = (\mu^2 + \mathbf{q}_1^2)^{1/2}$, and denote by l_β^{-1} the Lorentz transformation (in the xz-plane) which transforms the c.m. to rest, so that $p_1 = l_\beta p_1^0$, $q_1 = l_\beta q_1^0$, where $p_1^0 q_1^0$ is an initial state satisfying the c.m. condition. Then according to the transformation law of helicity states[4]

$$\mathfrak{L}_\beta \mid p_1^0\lambda\rangle = \sum_{\lambda'} U_{\lambda'\lambda}(p_1 ; l_\beta) \mid p_1\lambda'\rangle \tag{19}$$

where U is a spin-rotation matrix corresponding to a rotation about the y-axis. The rotation angle is indicated in the diagram in velocity space; see Fig. 2. Besides the velocity points corresponding to p_1 and q_1, the diagram has a point representing the velocity of the c.m. (a cross) and a point 0 representing the velocity of the arbitrary system, in which the momenta are p_1 and q_1. From Eq. (19) and the Lorentz invariance of the T-matrix one easily derives the connection

[3] See, for example, ref. *1*.
[4] See, for example, ref. *2*, especially the Appendix.

$$G_{\mu\lambda}(s,\, t) \;=\; \sum_{\mu'\lambda'} U_{\mu\mu'}(l_\beta^{-1};\, p_2) G_{\mu'\lambda'}(p_2 q_2\,;\, p_1 q_1) U_{\lambda'\lambda}(p_1\,;\, l_\beta). \tag{20}$$

A similar formula holds for an arbitrary Lorentz transformation l, except that in this case we would have on the left-hand side again a generalized amplitude for the values $l^{-1}p_2$, $\cdots$ etc. of the four-momenta.

We may perform the analytic continuations of $G_{\mu\lambda}(s, t)$ by continuing each of the three factors on the right hand side of Eq. (20). Let us assume, for the sake of simplicity, that the values (17) at the end of the trajectory satisfy the c.m. condition for the crossed reaction

$$\mathbf{q}_1 + \mathbf{Q}_2 = \mathbf{P}_1 + \mathbf{p}_2 = 0. \tag{21}$$

It then turns out that the end-value of the generalized helicity amplitude $G_{\mu'\lambda'}(p_2 q_2\,;\, p_1 q_1)$ coincides up to a phase factor (see Eq. (31)) with a helicity amplitude F for the crossed reaction. Such a relationship has in fact been conjectured by other authors (*3, 7*) with the difference that here, contrary to their results, the helicity does not change sign in the crossing process. Finally, the spin-rotation matrices U, or rather their analytic continuation, give rise to the orthogonal transformation Eq. (11), as has also been pointed out by Ya. Smorodinsky (*4*) and by Marinov and Roginskii (*3*).

Let us first examine the behavior of the generalized amplitude $G_{\mu\lambda}(p_2 q_2\,;\, p_1 q_1)$. We can do this in two ways. To begin with, we may say that G is given by Eq. (1) when the spinors are chosen to be "helicity-spinors" $u_\mu(p_2)$ and $u_\lambda(p_1)$. This means that, for example, $u(p_1)$ must satisfy, in addition to the Dirac equation $(i\gamma p_1 + m)u(p_1) = 0$ also a helicity condition[5]

$$\boldsymbol{\sigma}\cdot\mathbf{p}_1 u_\lambda(p_1) \;=\; 2\lambda(\mathbf{p}_1^{\,2})^{1/2} u_\lambda(p_1) \tag{22}$$

where $\lambda = \pm\frac{1}{2}$.

Since we have made the customary assumption that there is no problem in continuing the matrix T, Eq. (1) along a path such as that of Fig. 1, the whole question reduces to the behavior of the helicity spinors. Since p_2 reverts to the positive *real* (i.e., $\mathbf{p}_2$ real) mass shell in the end, it is not hard to see that the helicity μ remains unchanged in the process. The case of p_1 requires more care. At the end of the process, u will, of course, become a negative energy spinor, $v(P_1)$ satisfying a Dirac equation $(i\gamma P_1 - m)v(P_1) = 0$. If we assume, moreover, that along the trajectory $\mathbf{p}_1^{\,2}$ never becomes zero, then since Eq. (22) is always satisfied by analytic continuation, it follows that the final $v(P_1)$ must satisfy

[5] Equation (22) does not, of course, determine the phase and normalization of $u_\lambda(p_1)$. We assume that the latter are chosen according to the convention explained above when $\mathbf{p}_1$ is real. They are then determined for the other values by analytic continuation.

$$-\boldsymbol{\sigma}\cdot\mathbf{P}_1 v_\lambda(P_1) = 2\lambda(\mathbf{P}_1^2)^{1/2} v_\lambda(P_1) \tag{22'}$$

where the sign of the square-root on the right hand side is determined unambiguously by continuity along the path. As we shall see presently, the sign of the square-root is determined to be positive, whereupon it follows that the *antinucleon* state described by v_λ has helicity $+\lambda$, as one can see immediately by examining the charge-conjugate spinor $C\bar{v}$ (or by the more elementary hole-theory argument: a missing particle of spin $-\lambda$ in the $\mathbf{P}_1$-direction corresponds to an antiparticle of spin $+\lambda$ in the same direction). Thus the helicity *does not change sign* in the analytic continuation.

The crux of the argument, it will be seen, is the behavior of the square-root of $\mathbf{P}_1^2$. This is not trivial, since the vector $\mathbf{P}_1$ becomes complex along the path. Let, however, ϵ_1 be the time component of p_1 at any point of the trajectory so that $(\mathbf{P}_1^2)^{1/2}$ is the end value of

$$(\epsilon_1^2 - m^2)^{1/2}. \tag{23}$$

The whole question hinges on the path of the representative point for ϵ_1 in the complex ϵ_1-plane, Fig. 3. Now assume for simplicity that q_1 and p_2 are exactly real at the endpoint. For the initial value of p_1 we write instead $p_1 + i\eta$ where p_1 and η are real, and $n = (\eta_0, \mathbf{n})$ is infinitesimal. Thus $p_1^2 = m^2$ and $p_1\cdot\eta = 0$. Calculating in the c.m. system for $p_1 + q_1$, i.e., assuming $\mathbf{p}_1 = -\mathbf{q}_1$, the condition for $s = (p_1 + i\eta + q_1)^2$ to have a positive imaginary part is

$$0 < \eta\cdot q_1 = \eta_0\omega_1 - \mathbf{n}\cdot\mathbf{q}_1 = \eta_0\omega_1 + \mathbf{n}\cdot\mathbf{p}_1 = \eta_0(\omega_1 + \epsilon_1).$$

This implies $\eta_0 > 0$, i.e., the initial value of ϵ_1 is $\epsilon_1 + i\eta_0$; it has a small *positive* imaginary part.

Similarly the condition that $t = (p_1 - p_2)^2$ has a small positive imaginary part at the endpoint of the trajectory implies that the end-value of ϵ_1 has a negative imaginary part. Thus the endpoints are as indicated in Fig. 3, and the remaining question is whether the ϵ_1-trajectory cuts the real axis between $-m$ and $+m$ (as indicated in the figure) or not. If it does, the end-value of (23) is positive, as we have assumed. Now this can certainly be arranged, if we assume that the time components of the four-vectors p_1 and q_2 are everywhere complex along the trajectory except at the crossing point. Since the latter is inside the triangle

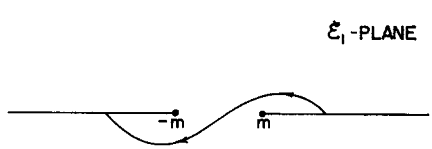

FIG. 3. The complex ϵ_1-plane

(9) we may assume that at this point the four-vectors $p_1, \cdots, q_2$ are all "Euclidean" (i.e., have a real time component and a pure imaginary space component). In this case the nucleon time components ϵ, and the meson time components ω satisfy the inequalities

$$-m < \epsilon < m, \qquad -\mu < \omega < +\mu, \tag{24}$$

so that all the energy trajectories satisfy the requirements (for mesons this would be of interest if they also had a spin).

In conclusion, the discussion above shows that, owing to the large number of variables, when analytic continuation in the four-momenta is involved, the answer is by no means unique,[6] but if one chooses the convention which can be stated in general in the most simple and natural way, then the result is the one we have indicated.

If one wishes to extend this conclusion to general spins and masses, one can resort to the second method we mentioned, namely, one generalizes (20) to an arbitrary Lorentz transformation l:

$$G_{\mu'\lambda'}(l^{-1}p_2 l^{-1}q_2\,;\,l^{-1}p_1 l^{-1}q_1) = \sum_{\mu\lambda} U_{\mu'\mu}(l^{-1};\,p_2)G_{\mu\lambda}(p_2q_2\,;\,p_1q_1)U_{\lambda\lambda'}(p_1\,;\,l) \tag{25}$$

(if all four particles have spins, there will be two more indices to G, and two more U matrices in the product). One then keeps l fixed, and continues analytically to the values (17). This requires analytic continuation of $U_{\lambda\lambda'}(p_1\,;\,l)$ to $U_{\lambda\lambda'}(-P_1\,;\,l)$. Let us write

$$U(p;\,l) = \mathfrak{D}\{h^{-1}(p)lh(l^{-1}p)\} \tag{26}$$

where $\mathfrak{D} \equiv \mathfrak{D}^{(s)}$ is the representation of the rotation group pertaining to the spin s of the particle. We have to write explicitly the components of the Lorentz transformations $h(p)$ and $h(p')$ as functions of the four components of p and $p' = l^{-1}p$, where $h(p)$ is defined according to the helicity convention

$$h(p) = r_{\phi,\theta,-\phi}z(p) \tag{27}$$

where θ, ϕ are the polar angles of $\mathbf{p}$; i.e. $h(p)$ is an ordinary Lorentz transformation of velocity $|\,\mathbf{p}\,|/p_0$ in the z-direction followed by rotations through Euler angles $-\phi$, θ, ϕ. One then performs the analytic continuation. In order to determine the transformation properties of the continuation of $G_{\mu\lambda}$, it is sufficient to consider small l in Eq. (25). In particular, we assume that p_{10} and, since l is small, also $(lp_1)_0$ cross the real axis between $+m$ and $-m$ as indicated in Fig. 3. With this assumption, it is straightforward to show that

$$(-1)^{\lambda-\lambda'}U_{\lambda\lambda'}(-P;\,l) = U_{\lambda'\lambda}(l^{-1};\,P) = U^{-1}_{\lambda'\lambda}(P;\,l) = U^{*}_{\lambda\lambda'}(P;\,l). \tag{28}$$

[6] We are indebted to Prof. R. Cutkosky for a remark in this connection.

The appearance of the complex conjugate is to be expected since the particle changes side of the reactions.

Equation (25) now becomes, for the end-values (17),

$$G_{\mu'\lambda'}(l^{-1}p_2, -l^{-1}Q_2; -l^{-1}P_1, l^{-1}q_1) = \sum_{\mu\lambda} (-1)^{\lambda-\lambda'} U_{\mu'\mu}(l^{-1}; p_2) \times U_{\lambda'\lambda}(l^{-1}; P_1) G_{\mu\lambda}(p_2, -Q_2; -P_1, q_1). \tag{29}$$

This transformation law is to be compared with that for the generalized helicity amplitude $F_{\mu\lambda}(p_2P_1; q_1Q_2)$. In order that this coincides with $F_{\mu\lambda}(s, t)$ as defined in (7) when $\mathbf{q}_1 + \mathbf{Q}_2 = 0$, the helicity state $| p_2, \mu \rangle$ must be defined with the additional factor $(-1)^{1/2-\mu}$. The transformation law is then

$$F_{\mu\lambda}(l^{-1}p_2, l^{-1}P_1; l^{-1}q_1, l^{-1}Q_2) = \sum_{\mu'\lambda'} (-1)^{\mu'-\mu} F_{\mu'\lambda'}(p_2, P_1; q_1, Q_2) \times U_{\mu'\mu}(l^{-1}; p_2) U_{\lambda'\lambda}(l^{-1}; P_1) \tag{30}$$

where the factor $(-1)^{\mu'-\mu}$ results from defining the nucleon helicity state with the factor $(-1)^{1/2-\mu}$. Thus, if there is a direct connection between F and G, it must be (apart from an over-all phase factor)

$$G_{\mu\lambda}(p_2, -Q_2; -P_1, q_1) = (-1)^{\mu-\lambda} F_{\mu\lambda}(p_2P_1; q_1Q_2). \tag{31}$$

(Note that if the antinucleon had been taken as "particle 2" there would be no factor $(-1)^{\mu-\lambda}$ in Eq. (31).) Notice also that no reversal of the sign of the helicity occurs; this circumstance is again strictly connected with a path such as indicated in Fig. 3. If we now specialize to a c.m. system $\mathbf{q}_1 + \mathbf{Q}_2 = 0$, the right hand side becomes $(-1)^{\mu-\lambda} F_{\mu\lambda}(s, t)$.

We now come to the final step, the analytic continuation of the U-factors in Eq. (20), which is different from the preceding case, because the Lorentz transformation l_β also varies along the path. We notice in fact that, along the path, l_β becomes a complex Lorentz transformation and at the end it becomes $l_{\beta'}$, where the velocity

$$\beta' = \frac{\mathbf{q}_1 - \mathbf{P}_1}{q_{10} - P_{10}} \tag{32}$$

may be greater than unity (i.e., than the velocity of light) so that $l_{\beta'}$ may also be complex. In particular, when the transformation is from an annihilation c.m. to an elastic scattering c.m., β' is infinite. Since the vectors are assumed to remain in the $x\,z$ plane, $U(p_1; l_\beta)$ and $U(l_\beta^{-1}; p_2)$ can each be expressed in terms of a single angle, χ_1 and χ_2, respectively. For the continuation process, we choose as the arbitrary point O of Fig. 2 the velocity point of the center of mass of p_2, P_1 or q_1, Q_2 reached at the end of the continuation. Let C denote the velocity point of the center of mass of p_1, q_1 or p_2, q_2. The angles may be determined

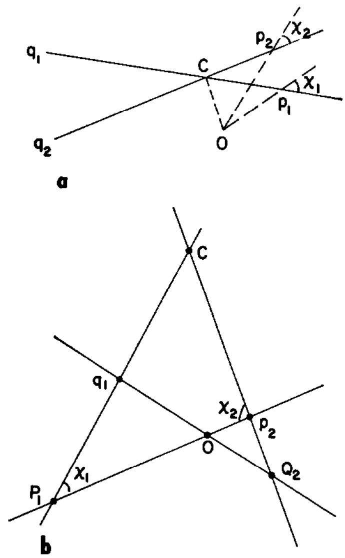

FIG. 4a. The velocity space diagram for an arbitrary point along the path of continuation. C represents the c.m. of p_1 and q_1 at this point; 0 represents the c.m. of P_1 and p_2 in the final configuration. b. The velocity space diagram for the final configuration.

from the cosine theorem (*2*) applied to the triangles OCp_1 and OCp_2 respectively; see Fig. 4a.

$$\cos\chi_1 = \frac{\cosh\rho_{C1}\cosh\rho_{01} - \cosh\rho_{0C}}{\sinh\rho_{C1}\sinh\rho_{01}},$$
$$\cos\chi_2 = \frac{\cosh\rho_{C2}\cosh\rho_{02} - \cosh\rho_{0C}}{\sinh\rho_{C2}\sinh\rho_{02}}, \tag{33}$$

where tanh ρ_{C1} is the absolute velocity of p_1 with respect to C, etc. For an arbitrary point on the path of continuation and using different masses for p_1 and p_2, q_1 and q_2 for the sake of generality

$$\cos\chi_1 = \frac{(s + m_1^2 - \mu_1^2)\,p_{10} - 2(p_{10} + q_{10})\,m_1^2}{S_1(p_{10}^2 - m_1^2)^{1/2}},$$
$$\cos\chi_2 = \frac{(s + m_2^2 - \mu_2^2)\,p_{20} - 2(p_{20} + q_{20})\,m_2^2}{S_2(p_{20}^2 - m_2^2)^{1/2}}, \tag{34}$$

evaluated in the rest system of O. Here $S_1^2 = [s - (m_1 - \mu_1)^2][s - (m_1 + \mu_1)^2]$, etc. At the end of the specified path, one obtains the positive determinations of S_i and $(p_{i0}^2 - m_i^2)^{1/2}$. Thus there are no new problems in the continuation of cos χ_i. We must next determine how sin χ_i continues. Consider the initial configuration when O and C are connected by a real Lorentz transformation. Let θ denote the angle from $-\mathbf{p}_1$ to $\mathbf{q}_1$ in 0. Further, define α_1 to be the angle from $\mathbf{p}_1 + \mathbf{q}_1$ to $\mathbf{p}_1$ in 0. (In order to determine signs correctly, it is essential to pay attention to the directions of the angles. For example, θ is positive if a positive rotation about the y-axis takes $-\mathbf{p}_1$ parallel to $\mathbf{q}_1$.) These two angles are related by

$$\sin \alpha_1 = \sin \theta (q_{10}^2 - \mu_1^2)^{1/2}/[(p_{10} + q_{10})^2 - s]^{1/2}, \tag{35}$$

where the positive determination of the square roots must be taken. The sine theorem (2) applied to triangle OCp_1, Fig. 4a, yields the relation

$$\sin \chi_1 = \sin \alpha_1 \frac{2m_1 [(p_{10} + q_{10})^2 - s)^{1/2}}{S_1}, \tag{36}$$

and again the positive determination of the square roots is to be taken. Consequently,

$$\sin \chi_1 = \sin \theta \frac{2m_1(q_{10}^2 - \mu_1^2)^{1/2}}{S_1}; \tag{37}$$

since the continuations of S_1 and $(q_{10}^2 - \mu^2)^{1/2}$ have already been specified, Eq. (37) allows us to express the continuation of $\sin \chi_1$ unambiguously in terms of $\sin \theta$. Clearly, $\sin \chi_2$ can be continued in the same way. This completes the continuation of Eq. (20) and provides the relation between $G_{\mu\lambda}(s, t)$ and $F_{\mu\lambda}(s, t)$.

Let us first apply these results to the πN problem. At the end of the continuation we have

$$\begin{aligned} \cos \chi_1 &= -\cos \chi_2 = -\frac{(s + m^2 - \mu^2)(t/4)^{1/2}}{S(t/4 - m^2)^{1/2}} = \frac{2E(p - q \cos \theta)}{S}, \\ \sin \chi_1 &= \sin \chi_2 = \frac{2mq \sin \theta}{S}. \end{aligned} \tag{38}$$

Equation (20) is then

$$G_{\mu\lambda}(s, t) = \sum_{\mu' \lambda'} d^{1/2}_{\mu'\mu}(\pi - \chi_1)\, d^{1/2}_{\lambda'\lambda}(\chi_1)(-1)^{\mu' - \lambda'} F_{\mu'\lambda'}(s, t), \tag{39}$$

or

$$G_{++} = \sin \chi_1 F_{++} + \cos \chi_1 F_{+-}, \qquad G_{+-} = \cos \chi_1 F_{++} - \sin \chi_1 F_{+-}, \tag{40}$$

which agrees with Eq. (11).

The formulas are readily generalized to the case where all four particles have spin. For notational convenience, we continue to write the formulas as if a baryon of mass m_1 and spin s_1 is crossed with a meson of mass μ_2 and spin σ_2. These are easily translated to other cases. The relation between the helicity amplitude is (baryon indices μ, λ, meson indices α, β):

$$G_{\mu\beta,\lambda\alpha}(s,t) = \sum_{\mu',\lambda',\beta',\alpha'} (-1)^{\eta}\, d^{s_2}_{\mu'\mu}(\chi_2)\, d^{s_1}_{\lambda'\lambda}(\chi_1)\, d^{\sigma_2}_{\beta'\beta}(\psi_2)\, d^{\sigma_1}_{\alpha'\alpha}(\psi_1)\, F_{\mu'\lambda',\beta'\alpha'}(s,t), \quad (41)$$

where

$$\cos\chi_1 = \frac{-(s+m_1^2-\mu_1^2)(t+m_1^2-m_2^2) - 2m_1^2(m_2^2-m_1^2+\mu_1^2-\mu_2^2)}{2p\sqrt{t}\,S_1},$$

$$\cos\chi_2 = \frac{(s+m_2^2-\mu_2^2)(t+m_2^2-m_1^2) - 2m_2^2(m_2^2-m_1^2+\mu_1^2-\mu_2^2)}{2p\sqrt{t}\,S_2},$$

$$\cos\psi_1 = \frac{(s+\mu_1^2-m_1^2)(t+\mu_1^2-\mu_2^2) - 2\mu_1^2(m_2^2-m_1^2+\mu_1^2-\mu_2^2)}{2q\sqrt{t}\,S_1}, \quad (42)$$

$$\cos\psi_2 = \frac{-(s+\mu_2^2-m_2^2)(t+\mu_2^2-\mu_1^2) - 2\mu_2^2(m_2^2-m_1^2+\mu_1^2-\mu_2^2)}{2q\sqrt{t}\,S_2},$$

and

$$\sin\chi_1 = 2m_1 q\sin\theta/S_1, \qquad \sin\psi_1 = 2\mu_1 p\sin\theta/S_1, \quad (43)$$
$$\sin\chi_2 = 2m_2 q\sin\theta/S_2, \qquad \sin\psi_2 = 2\mu_2 p\sin\theta/S_2,$$

with

$$p = [t^2 - 2t(m_1^2+m_2^2) + (m_1^2-m_2^2)^2]^{1/2}/2\sqrt{t},$$
$$q = [t^2 - 2t(\mu_1^2+\mu_2^2) + (\mu_1^2-\mu_2^2)^2]^{1/2}/2\sqrt{t}.$$

The quantity η depends on which particles are defined as "particle 2" in G and F. (See the discussion following Eqs. (2) and (31).) Suppose that in G the mesons are taken to be "particle 2". The value of η for the various possible choices in the definition of F are tabulated below:

"particle 2"	η
p_2, Q_2	$\lambda' - \mu' + \alpha' - \beta'$
p_2, q_1	$\lambda' - \mu'$
P_1, q_1	0
P_1, Q_2	$\alpha' - \beta'$.

Perhaps the easiest convention to remember is that $\eta = 0$ if an uncrossed particle is "particle 1" in both G and F while a crossed particle changes from "particle 1" to "particle 2" and vice versa.

APPENDIX

A short explanation of the expression for $\cos(\theta_s/2)$, Eq. (10), is given. First note that from (5) and (8)

$$\cos(\theta_s/2) = \pm 2ipq \sin\theta/S, \tag{A.1}$$

or

$$\sin\theta_s = \pm 4ipq(-st)^{1/2} \sin\theta/S^2, \tag{A.2}$$

for an arbitrary point on the path of continuation. Consider the invariant quantity

$$\Phi = \begin{vmatrix} q_{10} & q_{1x} & q_{1z} \\ p_{10} & p_{1x} & p_{1z} \\ p_{20} & p_{2x} & p_{2z} \end{vmatrix}.$$

We evaluate Φ in the initial configuration in the center of mass of p_1, q_1, continue the expression to the final configuration, and evaluate in the center of mass of P_1, p_2. Initially,

$$\Phi = \sqrt{s}(p_{1x}p_{2x} - p_{2z}p_{1z}) = -S^2 \sin\theta_s/4\sqrt{s}; \tag{A.4}$$

finally

$$\Phi = -\sqrt{t}(Q_{1x}P_{1z} - P_{1x}Q_{1z}) = -2Epq \sin\theta, \tag{A.5}$$

or

$$\begin{aligned} \sin\theta_s &= 8E\, pq\sqrt{s} \sin\theta/S^2 \\ &= -4ipq(-st)^{1/2} \sin\theta/S^2, \end{aligned} \tag{A.6}$$

and hence one must take the minus sign in (A.1). Note that this relation is independent of which way the path of continuation circles $us = (m^2 - \mu^2)^2$; in fact, the path of Fig. 1 and the condition that $\sin\theta_s > 0$ initially require $\sin\theta < 0$ in the final configuration.

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REFERENCES

1. M. JACOB AND G. C. WICK, *Ann. Phys.* (*N.Y.*) **7**, 404 (1959).
2. G. C. WICK, *Ann. Phys.* (*N.Y.*) **18**, 65 (1962).
3. M. S. MARINOV AND V. I. ROGINSKII, preprint from Institute for Experimental and Theoretical Physics, Moscow, 1963.
4. YA. A. SMORODINSKY, Dubna preprint, 1963.
5. G. F. CHEW, M. GOLDBERGER, F. LOW, AND Y. NAMBU, *Phys. Rev.* **106**, 1337 (1957).
6. W. R. FRAZER AND J. R. FULCO, *Phys. Rev.* **117**, 1603 (1960); *ibid.* **119**, 1420 (1960).
7. A. O. BARUT, *Phys. Rev.* **130**, 436 (1963).